Minicourse on resonances in hyperbolic dynamics

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Introduction

These are notes supporting a minicourse given during the summer school "Microlocal and probabilistic methods in geometry and dynamics" in Jussieu. The goal of this minicourse is to explain how spaces of anisotropic distributions can be used to prove an asymptotic expansion for the correlations of transitive Anosov diffeomorphisms (Theorem 1).

The spaces that we use are adapted from the ideas of [FR06] but the methods that we use to study the action of composition operators on these spaces rather follow the strategy from [BT07, BT08, Bal18] based on Paley–Littlewood decomposition.

1 Anosov diffeomorphisms

1.1 Definition and basic properties

Definition 1.1. Let M be a compact C^∞ manifold. Endow M with any smooth Riemannian metric. Let $F : M \to M$ be a C^1 diffeomorphism. We say that F is *Anosov* if, for every $x \in M$, there is a decomposition of the tangent space of M at x in $T_x M = E_x^u \oplus E_x^s$, such that the following properties hold:

- **invariance:** for every $x \in M$, we have $D_x F(E_x^u) = E_{Fx}^u$ and $D_x F(E_x^s) = E_{Fx}^s$;
- **hyperbolicity:** there are constants $C, \lambda > 1$ such that for every $x \in M, n \in \mathbb{N}, v_s \in E_x^s$ and $v_u \in E_x^u$, we have:

$$|D_x F^n \cdot v_s| \leq C\lambda^{-n}|v_s| \text{ and } |D_x F^{-n} \cdot v_u| \leq C\lambda^{-n}|v_u|.$$

Remark 1.2. Notice that we make no regularity assumption on the stable and unstable directions in Definition 1.1. However, one can prove from the definition that E_x^u and E_x^s depend continuously on x.

Example 1.3. Let $d \geq 1$. Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an invertible matrix with integer coefficients, whose inverse also has integer coefficients. The matrix A induces a diffeomorphism F of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, called a CAT map. Let us prove that if A has no eigenvalue of modulus 1, then F is Anosov.

Let E^u denote the sum of the characteristic spaces of A corresponding to eigenvalues of modulus greater than 1, and E^s the sum of the characteristic spaces of A corresponding to eigenvalues of modulus less than 1. Since A has no eigenvalue of modulus 1, we have $\mathbb{R}^d = E^u \oplus E^s$. Moreover, E^u and E^s are stable under the action of A, and the endomorphisms of E^u and E^s induced by A have spectral radius respectively strictly greater than 1 and strictly less than 1.

Now, using the standard parallelization $T\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$, the derivative of F is just the map $(x, v) \mapsto (Fx, Av)$. Thus, we can define for $x \in \mathbb{T}^d$ the stable and unstable directions by $E^u_x = \{x\} \times E^u$ and $E^s_x = \{x\} \times E^s$. The invariance and hyperbolicity properties follow then from the definition of E^u and E^s.

1.2 Stable and unstable manifolds

Proposition 1.4 (Local stable manifolds)**.** *Let M be a compact C^∞ manifold and F a C^∞ Anosov diffeomorphism on M. Let d be a distance associated to a Riemannian metric on M. Then there is $\epsilon > 0$ such that for every $x \in M$ the set*

$$W^s_\epsilon(x) = \{y \in M : d(F^n x, F^n y) < \epsilon \text{ for every } n \geq 0\}$$

is a C^∞ manifold, called the local stable manifold of F at x. *Moreover:*

- *for every $x \in M$ and $y \in W^s_\epsilon(x)$, we have $T_y W^s_\epsilon(x) = E^s_y$;*
- *there are $C > 0$ and $\theta \in (0, 1)$ such that for every $x \in M, y \in W^s_\epsilon(x)$ and $n \geq 0$ we have $d(F^n x, F^n y) \leq C\theta^n d(x, y)$;*
- *the manifold $W^s_\epsilon(x)$ depends smoothly on x.*

See for instance [Yoc95, §3.5] for a proof of Proposition 1.4 and the meaning of the third point. By replacing F by F^{-1}, one get a similar result for local unstable manifolds

$$W^u_\epsilon(x) = \left\{y \in M : d(F^{-n} x, F^{-n} y) < \epsilon \text{ for every } n \geq 0\right\} \text{ for } x \in M.$$

The global stable manifold of $x \in M$ may then be defined as

$$W^s(x) = \left\{ y \in M : d(F^n x, F^n y) \underset{n \to +\infty}{\to} 0 \right\}.$$

Notice then that if $y \in W^s(x)$ then $F^n(y) \in W^s_\epsilon(F^n x)$ for n large enough, and it follows that

$$W^s(x) = \bigcup_{n \geq 0} F^{-n}(W^s_\epsilon(F^n x)),$$

which implies that $W^s(x)$ is a C^∞ immersed submanifold of M. One defines similarly the global unstable manifold of x:

$$W^u(x) = \left\{ y \in M : d(F^{-n} x, F^{-n} y) \underset{n \to +\infty}{\to} 0 \right\} = \bigcup_{n \geq 0} F^n(W^u_\epsilon(F^{-n} x)).$$

Provided that $\epsilon > 0$ is small enough, for every $x \in M$ the manifolds $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ have a unique point of intersection, and this intersection is transverse. It follows that there is $\delta > 0$ such that for every $x, y \in M$ such that $d(x, y) \leq \delta$ the manifolds $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ have a unique point of intersection $[x, y]$. Moreover, the map $(x, y) \mapsto [x, y]$ is continuous from $\{x, y \in M; d(x, y) \leq \delta\}$ to M. Notice that $[x, y]$ is a point whose orbit is asymptotic to the orbit of x in the future and to the orbit of y in the past. We will use this construction in the proof of Lemma 4.6 below.

1.3 Asymptotics of correlations

The goal of this minicourse is to give a proof in a particular case of the following result.

Theorem 1 ([GL06])**.** *Let M be a compact C^∞ manifold and $F : M \to M$ be a C^∞ transitive Anosov diffeomorphism. There is a discrete bounded subset* Res *of $\mathbb{C} \setminus \{0\}$, and, for each $\lambda \in \mathrm{Res}$, a non-negative integer $N(\lambda)$, and $N(\lambda)+1$ non-trivial continuous bilinear forms $a_{\lambda,0}, \dots, a_{\lambda,N(\lambda)}$ from $C^\infty(M) \times C^\infty(M)$ to $\mathbb{C}$, such that for each $f, g \in C^\infty(M)$ and $\eta > 0$, we have*

$$\int_M f \circ F^n g \mathrm{d}x \underset{n \to +\infty}{=} \sum_{\substack{\lambda \in \mathrm{Res} \\ |\lambda| \geq \eta}} \sum_{k=0}^{N(\lambda)} a_{\lambda,k}(f,g) n^k \lambda^n + \mathcal{O}(\eta^n). \tag{1}$$

Remark 1.5.

1. To define the integral in the left hand side of (1), one can use any smooth positive density equivalent to Lebesgue on M.
2. The elements of Res are called the (Ruelle–Pollicott) resonances of F.
3. One can be slightly more precise, the bilinear forms $a_{\lambda,k}$'s from Theorem 1 factorize through finite dimensional spaces. Thus, Theorem 1 can be reformulated in the following way. For every $\eta > 0$, there are $D \geq 0$, a $D \times D$ matrix B and two continuous linear maps P, Q from $C^\infty(M)$ to $\mathbb{R}^D$ such that for every $f, g \in C^\infty(M)$ we have
$$\int_M f \circ F^n g \mathrm{d}x \underset{n \to +\infty}{=} \langle B^n P f, Q g \rangle + \mathcal{O}(\eta^n).$$
4. The $\mathcal{O}(\eta^n)$ in (1) is controlled by the C^k norms of f and g for some k that depends on η.

Remark 1.6. Notice that the set Res, the numbers $N(\lambda)$ and the coefficients $a_{\lambda,k}$ are uniquely determined by the sequence $\left(\int_M f \circ F^n g \mathrm{d}x\right)_{n \geq 0}$. Indeed, consider the function

$$\Psi_{f,g}(z) = \sum_{n \geq 0} z^{-(n+1)} \int_M f \circ F^n g \mathrm{d}x$$

which is holomorphic for $|z| \gg 1$. Using (1), we find that $\Psi_{f,g}(z)$ has a meromorphic continuation to $\mathbb{C} \setminus \{0\}$ whose poles are contained within Res. Moreover, if $\lambda \in \mathrm{Res}$, we have near λ

$$\Psi_{f,g}(z) = H(z) + \sum_{k=0}^{N(\lambda)} \frac{c_{\lambda,k}(f,g)}{(z-\lambda)^{k+1}},$$

where the function $H(z)$ is holomorphic near λ. Moreover, we have $c_{\lambda,k}(f,g) = k! \lambda^k a_{\lambda,k}(f,g) + G_{\lambda,k}(a_{\lambda,k+1}(f,g), \dots, a_{\lambda,N(\lambda)}(f,g))$ for $k = 0, \dots, N(\lambda)$, where the number $G_{\lambda,k}(a_{\lambda,k+1}(f,g), \dots, a_{\lambda,N(\lambda)}(f,g))$ is a linear combination of the coefficients $a_{\lambda,k+1}(f,g), \dots, a_{\lambda,N(\lambda)}(f,g)$. Thus, $\Psi_{f,g}(z)$ uniquely determines the $a_{\lambda,k}(f,g)$'s. Moreover, by taking generic f and g, all the $a_{\lambda,k}(f,g)$'s are non-zero. Consequently, when f and g run over $C^\infty(M)$, the meromorphic functions $\Psi_{f,g}(z)$ determine the set Res and the numbers $N(\lambda)$.

1.4 SRB measure

If F is a transitive C^∞ diffeomorphism on a compact manifold M, then one can associate to F a unique Sinaï–Ruelle–Bowen (SRB) measure μ. This is a Borel probability measure on M, invariant by F, which plays a particular role for the statistical properties of F.

There are several properties that (individually) distinguish the SRB measure μ among all F-invariant Borel probability measures:

- μ is *physical*: for Lebesgue almost every $x \in M$, we have for all continuous function $f : M \to \mathbb{C}$

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ F^k(x) \underset{n \to +\infty}{\to} \int_M f \mathrm{d}\mu.$$

- μ has smooth conditionals on local unstable manifolds (and thus is an equality in Ruelle's inequality).
- μ is the limit of invariant measures of certain small stochastic perturbations of F.
- The wave front set of μ is contained in $\left\{(x,\xi) \in T^*M : \xi_{|E_u} = 0\right\}$.

For the first three characterizations of μ, one can refer to [You02] and references therein. The last point follows for instance from [FRS08, Corollary 1]. Theorem 1 allows us to add a characterization of μ to this list.

Theorem 2 ([GL06])**.** *Under the assumption of Theorem 1, and if M is connected, the number* 1 *belongs to* Res*. Moreover, there is no other resonance of modules larger than or equal to* 1*,* $N(1) = 0$ *and*

$$a_{1,0} : (f,g) \mapsto \int_M f \mathrm{d}\mu \int_M g \mathrm{d}x.$$

Corollary 1.7. *Under the assumption of Theorem 1, and if M is connected, for every $f, g \in C^\infty(M)$, we have*

$$\int_M f \circ F^n g \mathrm{d}x \underset{n \to +\infty}{\to} \int_M f \mathrm{d}\mu \int_M g \mathrm{d}x. \tag{2}$$

Moreover, the convergence is exponentially fast.

Remark 1.8. Notice that the convergence (2) also follows from the physicality of μ and the dominated convergence theorem.

2 Asymptotics of correlations in the linear case

2.1 The doubling map

Before starting the proof of Theorem 1, let us study a toy-model: the doubling map of the circle. This is the map F on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ defined by $F(x) = 2x \mod 1$.

Let f and g be C^∞ functions on $\mathbb{S}^1$. As f and g are smooth, they are the sums of their Fourier series (the convergence actually holds in C^∞)

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{2i\pi kx} \text{ and } g(x) = \sum_{k\in\mathbb{Z}} c_k(g) e^{2i\pi kx} \text{ for } x \in \mathbb{S}^1,$$

where our convention for Fourier coefficients is

$$c_k(f) = \int_{\mathbb{S}^1} e^{-2i\pi kx} f(x)\mathrm{d}x \text{ for } k \in \mathbb{Z}.$$

Then, notice that for every $n \geq 0$, we have

$$f \circ F^n(x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{2i\pi 2^n kx} \text{ for } x \in \mathbb{S}^1.$$

It follows that

$$\int_{\mathbb{S}^1} f \circ F^n g \mathrm{d}x = \sum_{k\in\mathbb{Z}} c_k(f) c_{-2^n k}(g).$$

Now, take $N \geq 1$. Since f and g are C^∞, we find that there is a constant $C_N > 0$ such that for every $k \in \mathbb{Z} \setminus \{0\}$ we have

$$|c_k(f)| \leq C_N |k|^{-N} \text{ and } |c_k(g)| \leq C_N |k|^{-N}.$$

Thus, we have

$$\left| \int_{\mathbb{S}^1} f \circ F^n g \mathrm{d}x - \int_{\mathbb{S}^1} f \mathrm{d}x \int_{\mathbb{S}^1} g \mathrm{d}x \right| \leq 2C_N^2 2^{-nN} \sum_{k=1}^{+\infty} |k|^{-2N}.$$

Since N can be chosen arbitrarimy large, we get indeed an asymptotic expansion as in (1), with 1 being the only resonance.

2.2 CAT map

Let us now consider a particular case in Theorem 1. Let $A \in \mathrm{GL}(2,\mathbb{Z})$ be a matrix with no eigenvalue of modulus 1 and F be the associated CAT map (see Example 1.3). We want to adapt the argument in the previous section to prove Theorem 1 in this particular case.

Since A has no eigenvalues of modulus 1, so does ${}^tA^{-1}$. Let e_u^* and e_s^* be eigenvectors for ${}^tA^{-1}$ with eigenvalues respectively $\lambda, \pm\lambda^{-1}$ such that $|\lambda| > 1$. Define the cone

$$\mathcal{C}_u^* = \{ w_s e_s^* + w_u e_u^* : w_u, w_s \in \mathbb{R}, |w_s| \leq |w_u| \}. \tag{3}$$

Let us point out the following property of the cone $\mathcal{C}_u^*$:

$${}^tA^{-1}(\mathcal{C}_u^*) \subseteq \mathrm{Int}(\mathcal{C}_u^*) \cup \{0\}. \tag{4}$$

Moreover, there is a constant $C_0 > 0$ such that, for every $w \in \mathcal{C}_u^*$ and $n \geq 0$, we have

$$|{}^tA^{-n} w| \geq C_0^{-1} |\lambda|^n |w|, \tag{5}$$

and, for every $w \in \mathbb{R}^2 \setminus \mathcal{C}_u^*$ and $n \geq 0$ we have

$$|{}^t A^n w| \geq C_0^{-1} |\lambda|^n |w|. \tag{6}$$

To prove these properties, just use that the norm $w_u e_u^* + w_s e_s^* \mapsto |w_u| + |w_s|$ is equivalent to the standard norm on $\mathbb{R}^2$.

Now, let f, g be C^∞ functions on $\mathbb{T}^2$. As above, we decompose f and g in Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^2} c_k(f) e^{2\pi k \cdot x} \text{ and } g(x) = \sum_{k \in \mathbb{Z}^2} c_k(g) e^{2i\pi k \cdot x}$$

It follows that for $n \geq 0$

$$\begin{aligned} f \circ F^n(x) &= \sum_{k \in \mathbb{Z}^2} c_k(f) e^{2i\pi k \cdot A^n x} = \sum_{k \in \mathbb{Z}^2} c_k(f) e^{2i\pi {}^t A^n k \cdot x} \\ &= \sum_{k \in \mathbb{Z}^2} c_{{}^t A^{-n} k}(f) e^{2i\pi k \cdot x} \end{aligned}$$

Thus

$$\int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}x = \sum_{k \in \mathbb{Z}^2} c_{{}^t A^{-n} k}(f) c_{-k}(g).$$

Let $N \geq 3$. Since f and g are C^∞, there is a constant C_N such that

$$|c_k(f)| \leq C_N |k|^{-2N} \text{ and } |c_k(g)| \leq C_N |k|^{-N}.$$

Thus, we have

$$\left| \int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}x - \int_{\mathbb{T}^2} f \mathrm{d}x \int_{\mathbb{T}^2} g \mathrm{d}x \right| \leq C_N^2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-N} (|k| |{}^t A^{-n} k|)^{-N}. \tag{7}$$

For $k \in \mathbb{Z}^2 \setminus \{0\}$, let $T = \min \left\{ l \in \{0, \dots, n-1\} : {}^t A^{-\ell} k \in \mathcal{C}_*^u \right\}$, with the convention that $T = n$ if ${}^t A^{-(n-1)} k \notin \mathcal{C}_*^u$. Using (5), we find that if $T \neq n$ then

$$|{}^t A^{-n} k| = |{}^t A^{-(n-T)} ({}^t A^{-T} k)| \geq C_0^{-1} |\lambda|^{n-T} |{}^t A^{-T} k|. \tag{8}$$

Notice that this estimate still holds when $T = n$ (up to making C_0 larger). Using (6), we find that if $T \neq 0$ then

$$|k| \geq C_0^{-1} |\lambda|^{T-1} |{}^t A^{-(T-1)} k| \geq \widetilde{C}_0^{-1} |\lambda|^T |{}^t A^{-T} k|, \tag{9}$$

for some constant $\widetilde{C}_0 > 1$. Once again, this estimate clearly still holds when $T = 0$. Putting (8) and (9) together, we find that

$$|k| |{}^t A^{-n} k| \geq (C_0 \widetilde{C}_0)^{-1} |\lambda|^n |{}^t A^{-T} k|^2 \geq (C_0 \widetilde{C}_0)^{-1} |\lambda|^n,$$

where we used that ${}^t A^{-T} k$ has integer coefficients to find that $|{}^t A^{-T} k| \geq 1$.

Recalling (7), we find that

$$\left| \int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}x - \int_{\mathbb{T}^2} f \mathrm{d}x \int_{\mathbb{T}^2} g \mathrm{d}x \right| \leq C_N^2 C_0^N \widetilde{C}_0^N |\lambda|^{-nN} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-N}.$$

Since N can be chosen arbitrarily large and $|\lambda| > 1$, this estimate proves Theorem 1 in the case of a cat map on the torus $\mathbb{T}^2$. Notice that in that case, the resonance 1, whose existence is guaranteed by Theorem 2, is the only one. Moreover, Lebesgue measure on the torus is the SRB measure.

3 Asymptotics of correlations

3.1 Small perturbation of a CAT map

Let $A \in \mathrm{GL}(2,\mathbb{Z})$. Assume that A has no eigenvalue of modulus 1 and let F_0 be the associated CAT map. In this minicourse, we will prove Theorem 1 for F of the form $F = F_0 + \varphi$ where $\varphi : \mathbb{T}^2 \to \mathbb{R}^2$ is a C^∞ function with $\|\varphi\|_{C^1} \ll 1$. Even if we will not need it in our proof, we will prove that F is Anosov. This proof can be generalized to check that the Anosov property is C^1 open.

Proposition 3.1. *There is $\epsilon > 0$ such that for every C^∞ function $\varphi : \mathbb{T}^2 \to \mathbb{R}^2$ with $\|\varphi\|_{C^1} \leq \epsilon$ the map $F = F_0 + \varphi$ is Anosov.*

Proof. Let e_u and e_s be eigenvectors for A with associated eigenvalues respectively λ and $\pm\lambda^{-1}$ such that $|\lambda| > 1$. For $v \in \mathbb{R}^2$, write $v = v_u e_u + v_s e_s$. Define then the cones

$$\mathfrak{C}_u = \left\{ v_s e_s + v_u e_u : v_s, v_u \in \mathbb{R}, |v_s| \leq \frac{1}{2}|v_u| \right\}$$

and

$$\mathfrak{C}_s = \left\{ v_s e_s + v_u e_u : v_s, v_u \in \mathbb{R}, |v_u| \leq \frac{1}{2}|v_s| \right\}.$$

Notice then that $A(\mathfrak{C}_u) \subseteq \mathrm{Int}(\mathfrak{C}_u) \cup \{0\}$ and $A^{-1}(\mathfrak{C}_s) \subseteq \mathrm{Int}(\mathfrak{C}_s) \cup \{0\}$. Moreover, there is a constant $C > 0$ such that for every $n \geq 0, v \in \mathfrak{C}_u$ and $w \in \mathfrak{C}_s$ we have

$$|A^n v| \geq C^{-1}|\lambda|^n |v| \text{ and } |A^{-n} w| \geq C^{-1}|\lambda|^n |w|.$$

To prove these estimates, just use that the norm $v_s e_s + v_u e_u \mapsto |v_u| + |v_s|$ is equivalent to the standard norm on $\mathbb{R}^2$.

Choose then $\rho \in (1, |\lambda|)$ and n_0 large enough so that $C^{-1}|\lambda|^{n_0} > \rho^{n_0}$. Notice then that if ϵ is small enough and $\|\varphi\|_{C^1} \leq \epsilon$ then for every $x \in \mathbb{T}^2$ we have (we identify the derivative of F with an endomorphism of $\mathbb{R}^2$ through the standard parallelization)

$$D_x F(\mathfrak{C}_u) \subseteq \mathrm{Int}(\mathfrak{C}_u) \cup \{0\},$$
$$D_x F^{-1}(\mathfrak{C}_s) \subseteq \mathrm{Int}(\mathfrak{C}_s) \cup \{0\},$$

and for every $v \in \mathfrak{C}_u$ and $w \in \mathfrak{C}_s$ we have

$$|D_x F^{n_0} v| \geq \rho^{n_0}|v| \text{ and } |D_x F^{-n_0} w| \geq \rho^{n_0}|w|.$$

It implies that there is a constant $C > 0$ such that for every $n \geq 0, v \in \mathfrak{C}_u$ and $w \in \mathfrak{C}_s$ we have

$$|D_x F^n v| \geq C^{-1}\rho^n |v| \text{ and } |D_x F^{-n} w| \geq C^{-1}\rho^n |w|. \tag{10}$$

We are now ready to construct the unstable direction for F. For each $x \in \mathbb{T}^2$ and $n \geq 0$, write

$$D_{F^{-n}x} F^n \cdot e_u = p_n(x) e_u + q_n(x) e_s.$$

Since $e_u \in \mathfrak{C}_u$, so does $p_n(x) e_u + q_n(x) e_s$ and thus there is some new constant C (that does not depend on n nor x) such that $|p_n(x)| \geq C^{-1}\rho^n$ (here we use (10)). Define $a_n(x) = q_n(x)/p_n(x)$, and notice that

$$(a_n(x) - a_{n+1}(x)) D_x F^{-n} \cdot e_s = \frac{e_u}{p_n(x)} - \frac{D_{F^{-(n+1)}x} F \cdot e_u}{p_{n+1}(x)}.$$

Using (10) again, we find that there is a constant $C > 0$ that does not depend on n nor x such that $|a_n(x) - a_{n+1}(x)| \leq C\rho^{-2n}$. It follows that the sequence $(a_n(x))_{n\geq 0}$ has a limit $a(x)$.

Notice then that for $x \in \mathbb{T}^2$ and $n \geq 0$ we have

$$\begin{aligned} D_x F \cdot (e_u + a_n(x)e_s) &= D_x F \cdot \left(\frac{D_{F^{-n}x} F^n \cdot e_u}{p_n(x)} \right) \\ &= \frac{p_{n+1}(Fx)e_u + q_{n+1}(Fx)}{p_n(x)} \\ &= \frac{p_{n+1}(Fx)}{p_n(x)}(e_u + a_{n+1}(Fx)e_s). \end{aligned}$$

Since $(a_n(x))_{n\geq 0}$ and $(a_{n+1}(Fx))_{n\geq 0}$ have limits, it follows that the sequence $(p_{n+1}(Fx)/p_n(x))_{n\geq 0}$ has a limit, call it $\lambda_F(x)$, that satisfies

$$D_x F \cdot (e_u + a(x)e_s) = \lambda_F(x)(e_u + a(Fx)e_s). \tag{11}$$

For $x \in \mathbb{T}^2$, we let E^u_x be the line generated by $e_u + a(x)e_s$. It follows from (11) that $D_x F(E^u_x) = E^u_{Fx}$. Notice also that $E^u_x \subseteq \mathfrak{C}_u$ (the cone is closed), and thus it follows from (10) that the vectors of E^u_x are uniformly expanded in the sense of Definition 1.1.

The stable direction is constructed similarly replacing F by F^{-1}. □

3.2 Hilbert space of anisotropic distributions

For $\alpha \in \mathbb{R}$ and $f \in \mathcal{D}'(\mathbb{T}^2)$, define the norm

$$\|f\|^2_\alpha := \sum_{\substack{k\in\mathbb{Z}^2 \\ k\in\mathcal{C}^u_*}} (1+|k|^2)^\alpha |c_k(f)|^2 + \sum_{\substack{k\in\mathbb{Z}^2 \\ k\notin\mathcal{C}^u_*}} (1+|k|^2)^{-\alpha} |c_k(f)|^2. \tag{12}$$

Define then the space

$$\mathcal{H}_\alpha = \left\{ f \in \mathcal{D}'(\mathbb{T}^2) : \|f\|_\alpha < \infty \right\}.$$

Due to the different weights in the different direction in the norm (12), the elements of the space $\mathcal{H}_\alpha$ have regularity properties that depend on the direction in the physical space (see Proposition 4.1 below). For this reason, one sometimes calls $\mathcal{H}_\alpha$ a space of anisotropic distributions or an anisotropic space of distribution.

The following properties of $\mathcal{H}_\alpha$ are easily proven:

- $\mathcal{H}_\alpha$ is a Hilbert space.
- $C^\infty(M)$ is a dense subspace of $\mathcal{H}_\alpha$.
- $H^{-|\alpha|} \subseteq \mathcal{H}_\alpha \subseteq H^{|\alpha|}$ with continuous inclusions (where for $t \in \mathbb{R}$, we write H^t for the standard Sobolev space).
- The L^2 pairing induces an isomorphism between $\mathcal{H}_{-\alpha}$ and the dual of $\mathcal{H}_\alpha$.

We will use an alternative definition of the space $\mathcal{H}_\alpha$, in the spirit of the Paley–Littlewood decomposition (following the approach from [BT07, BT08, Bal18]). For $f \in \mathcal{D}'(\mathbb{T}^2)$, define

$$\Pi_0^u f = 0 \text{ and } \Pi_0^s f = c_0(f), \tag{13}$$

and for $N \geq 1$

$$\Pi_N^u f(x) = \sum_{\substack{k \in \mathbb{Z}^2 \\ k \in \mathcal{C}_u^* \\ 2^{N-1} \leq |k| < 2^N}} c_k(f) e^{2i\pi k \cdot x} \text{ and } \Pi_N^s f(x) = \sum_{\substack{k \in \mathbb{Z}^2 \\ k \notin \mathcal{C}_u^* \\ 2^{N-1} \leq |k| < 2^N}} c_k(f) e^{2i\pi k \cdot x}. \tag{14}$$

Notice then that

$$f = \sum_{N \geq 0} \Pi_N^u f + \sum_{N \geq 0} \Pi_N^s f \tag{15}$$

in $\mathcal{D}'(\mathbb{T}^2)$. Rather than (12), we will use the following equivalent norm on $\mathcal{H}_\alpha$:

$$\|f\|_\alpha^2 = \sum_{N \geq 0} 4^{\alpha N} \|\Pi_N^u f\|_{L^2}^2 + \sum_{N \geq 0} 4^{-\alpha N} \|\Pi_N^s f\|_{L^2}^2 .$$

Let us also recall that for every $L \in \mathbb{R}$ the Sobolev space H^L can be defined as the space of distributions f on $\mathbb{T}^2$ such that the norm

$$\|f\|_{H^L}^2 \coloneqq \sum_{n \geq 0} 4^{LN} (\|\Pi_N^u f\|_{L^2}^2 + \|\Pi_N^s f\|_{L^2}^2)$$

is finite. This norm is equivalent to the standarn norm on H^L.

3.3 Lasota–Yorke inequality

The proof of Theorem 1 is based on the study of the *Koopman operator* $\mathcal{K}$ associated to F defined by

$$\mathcal{K} f = f \circ F \text{ for } f \in \mathcal{D}'(M).$$

Notice that references sometimes study instead its formal adjoint, the *(Ruelle–Perron–Frobenius) transfer operator* defined by

$$\mathcal{L} f = \frac{1}{|\det DF| \circ F^{-1}} f \circ F^{-1} \text{ for } f \in \mathcal{D}'(M).$$

As mentioned above, we will prove Theorem 1 for the examples of Anosov diffeomorphisms from §3.1. The main technical result behind the proof of Theorem 1 is the following Lasota–Yorke inequality. With Lemma 3.2, the proof of Theorem 1 is reduced to spectral theoretic considerations that are exposed in §3.4.

Lemma 3.2 (Lasota–Yorke inequality)**.** *Under the assumption of §3.1, there is* $\tau \in (0,1)$ *and* $\epsilon > 0$ *such that if* $\|\varphi\|_{C^1} \leq \epsilon, \alpha \geq \epsilon^{-1}$*, then* $\mathcal{K}$ *induces a bounded operator on* $\mathcal{H}_\alpha$*. Moreover, for every* $L \in \mathbb{R}$*, there is a constant* $C > 0$ *such that for every* $n \in \mathbb{N}$*, there is a constant* $C_{n,L}$ *such that for every* $f \in \mathcal{H}_\alpha$ *we have*

$$\|\mathcal{K}^n f\|_\alpha \leq C \tau^{\alpha n} \|f\|_\alpha + C_{n,L} \|f\|_{H^L} . \tag{16}$$

Let us explain how we will use the fact that $\|\varphi\|_{C^1}$ is small in the proof of Lemma 3.2. Recall the properties (4), (5) and (6) of the matrix A. Let $\rho \in (1, |\lambda|)$ and notice that if $\|\varphi\|_{C^1}$ is small enough then there is a constant $C_0 \geq 1$ such that for every $n \geq 0, x \in \mathbb{T}^2, v \in \mathcal{C}_*^u$ and $w \in \mathbb{R}^2 \setminus \mathcal{C}_*^u$ we have

$$ {}^t(D_x F)^{-1}(\mathcal{C}_*^u) \subseteq \mathrm{Int}(\mathcal{C}_*^u) \cup \{0\}, \tag{17} $$

$$ |{}^t(D_x F^n)^{-1} v| \geq C_0^{-1} \rho^n |v| \text{ and } |{}^t(D_x F^{-n})^{-1} w| \geq C_0^{-1} \rho^n |w|. \tag{18} $$

With these properties in mind we want to adapt the argument from §2.2. However, in the non-linear case the action of $\mathcal{K}$ on Fourier coefficients is more involved. The main idea is that the frequencies transition that are away from those that appear in the linear case will only contribute to (16) through the compact term $\|f\|_{H^L}$. The actual proof of Lemma 3.2 presented here is inspired from [BT07, BT08, Bal18].

In order to distinguish the frequencies transition that are allowed or not, let $\Gamma = \mathbb{N} \times \{u, s\}$ and for each $n \geq 0$ introduce a relation $\hookrightarrow_n$ on Γ by $(N, t) \hookrightarrow (N', t')$ if one of the following properties hold:

- $t = t' = u$ and $N \leq N' - n \log_2 \rho + \log_2 C_0 + 2$;
- $t = t' = s$ and $N' \leq N - n \log_2 \rho + \log_2 C_0 + 2$;
- $t = s, t' = u$ and $N \geq n \log_2 \rho$ or $N' \geq n \log_2 \rho$.

Notice that the relation $\hookrightarrow_n$ select the frequencies transitions close to the linear case, and all the transitions that are "more favorable". In order to get rid of the other frequencies transition, we use the non-stationary phase method to prove the following lemma.

Lemma 3.3. *Let $\|\varphi\|_{C^1}$ be small enough so that (17) and (18) hold. For every $n \geq 1$ and every $\beta \geq 0$, there is a constant $C_{n,L} > 0$ such that if $(N, t) \not\hookrightarrow_n (N', t')$ then $\left\| \Pi_N^t \mathcal{K}^n \Pi_{N'}^{t'} \right\|_{L^2 \to L^2} \leq C_{n,L} 2^{-\beta \max(N, N')}$.*

Proof. Let $(N, t), (N', t') \in \Gamma$ be such that $(N, t) \not\hookrightarrow_n (N', t')$.

Notice that if $\max(N, N') \leq n \log_2 \rho + 1$ then the result follows by taking $C_{n,L}$ large enough (it corresponds to a finite number of case). Consequently, we will henceforth assume that $\max(N, N') \geq n \log_2 \rho + 1$. Notice that this reduction gets rid of the case $(t, t') = (s, u)$. It also implies in the case $t = t' = u$ that $N \neq 0$ and in the case $t = t' = s$ that $N' \neq 0$.

Let $k, k' \in \mathbb{Z}^2$ be indexes that appear in the sum defining Π_N^t and $\Pi_{N'}^{t'}$ respectively (recall (13) and (14)). Let us write the Fourier coefficient

$$ c_k(\mathcal{K}^n e^{2i\pi k' \cdot x}) = \int_{\mathbb{T}^2} e^{2i\pi \Phi_{k,k'}(x)} \mathrm{d}x, $$

where the phase $\Phi_{k,k'}$ is defined on the torus by

$$ \Phi_{k,k'} : x \mapsto k \cdot x - k' \cdot F^n x. $$

We are mostly interested in the gradient of this phase.

$$ \nabla \Phi_{k,k'}(x) = k - {}^t D_x F^n \cdot k' \text{ for } x \in \mathbb{T}^2. $$

We claim that the property $(N,t) \not\to_n (N',t')$ implies that for every $x \in \mathbb{T}^2$ we have

$$|\nabla \Phi_{k,k'}(x)| \geq C_n^{-1} 2^{\max(N,N')}, \tag{19}$$

where the constant C_n does not depend on N, N', t, t'.

Let us start with the case $t = t' = u$. In that case, we know that N is non-zero and thus $|k| \geq 2^{N-1}$. Hence, for some $C > 0$, we have

$$\begin{aligned}
|\nabla \Phi_{k,k'}(x)| &\geq C^{-1} |{}^t(D_x F^n)^{-1} \cdot k - k'| \\
&\geq C^{-1}(|{}^t(D_x F^n)^{-1} \cdot k| - |k'|) \geq C^{-1}(C_0^{-1} \rho^n |k| - |k'|) \\
&\geq C^{-1}(C_0^{-1} \rho^n 2^{N-1} - 2^{N'}) \\
&\geq C^{-1} C_0^{-1} \rho^n 2^{N-2} \\
&\geq \widetilde{C}^{-1} 2^{\max(N,N')}.
\end{aligned}$$

Here, we used (18) in the second line and the fact that $N \geq N' - n \log_2 \rho + \log_2 C_0 + 2$ on the fourth and fifth line.

We proved (19) when $t = t' = u$. The case $t = t' = s$ is similar, and we got rid of the case $(t,t') = (s,u)$. Thus, we are left with the case $(t,t') = (u,s)$. Remember that we have

$$|\nabla \Phi_{k,k'}(x)| \geq C^{-1} |{}^t(D_x F^n)^{-1} \cdot k - k'|.$$

Moreover, it follows from (17) that ${}^t(D_x F^n)^{-1} \cdot k$ belongs to a closed cone within $\mathrm{Int}(\mathcal{C}_*^u) \cup \{0\}$. Since $k' \notin \mathcal{C}_*^u$, we find that the distance between the projections to the unit circle of ${}^t(D_x F^n)^{-1} \cdot k$ and k' is uniformly bounded below. Since at least one of them is bounded away from zero, we find that the distance between ${}^t(D_x F^n)^{-1} \cdot k$ and k' has the order of magnitude $2^{\max(N,N')}$, and (19) follows.

Let us introduce the differential operator L on $\mathbb{T}^2$ defined by

$$L f(x) = \frac{1}{2i\pi} \frac{\nabla f(x) \cdot \nabla \Phi_{k,k'}(x)}{|\nabla \Phi_{k,k'}(x)|^2}.$$

Since $L(e^{2i\pi \Phi_{k,k'}(x)}) = e^{2i\pi \Phi_{k,k'}(x)}$, we find that for $m \geq 0$, we have

$$c_k(\mathcal{K}^n e^{2i\pi k' x}) = \int_{\mathbb{T}^2} e^{2i\pi \Phi_{k,k'}(x)} ({}^t L)^m (1) \mathrm{d}x,$$

where ${}^t L$ denotes the formal adjoint of L (obtained by integrating by parts). By induction, we find that $({}^t L)^m(1)$ is a linear combination of terms of the form

$$\frac{\text{product of } \ell \text{ derivatives of } \Phi_{k,k'}}{|\nabla \Phi_{k,k'}|^{\ell+m}}.$$

Thus, it follows from (19) that

$$\left| c_k(\mathcal{K}^n e^{2i\pi k' x}) \right| \leq C_{m,n} 2^{-m \max(N,N')}.$$

Finally, notice that if $f \in L^2$ then

$$\begin{aligned}
\left\| \Pi_N^t \mathcal{K}^n \Pi_{N'}^{t'} f \right\|_{L^2} &\leq \sum_{k,k'} \left| c_k(\mathcal{K}^n e^{2i\pi k' \cdot x}) \right| |c_{k'}(f)| \\
&\leq \widetilde{C}_{m,n} 2^{-(m-2)\max(N,N')} \|f\|_{L^2},
\end{aligned}$$

where the sum is over the indexes k and k' that appear in the definition of Π_N^t and $\Pi_{N'}^{t'}$ respetively. We used that the number of terms in the sum is bounded by some constant times $2^{N+N'}$. The result follows by taking $m = \beta + 2$.

$\square$

Proof of Lemma 3.2. Assume that $\|\varphi\|_{C^1}$ is small enough so that (17) and (18) (and thus Lemma 3.3) hold. Without loss of generality, we may assume that $L < 0$.

Let $n \geq 1$ and $f \in \mathcal{H}_\alpha$. Start by estimating for $N \geq 0$

$$\begin{aligned}
\|\Pi_N^u \mathcal{K}^n f\|_{L^2}^2 &= \left\| \Pi_N^u \mathcal{K}^n \left(\sum_{N' \geq 0} \Pi_{N'}^u f + \sum_{N' \geq 0} \Pi_{N'}^s f \right) \right\|_{L^2}^2 \\
&\leq 2 \left\| \Pi_N^u \mathcal{K}^n \sum_{N' \geq N + n \log_2 \rho - \log_2 C_0 - 2} \Pi_{N'}^u f \right\|_{L^2}^2 \\
&\qquad + 2 \left\| \Pi_N^u \mathcal{K}^n \sum_{\substack{N' \geq 0, t \in \{u,s\} \\ (N,u) \not\leftrightarrow (N',t)}} \Pi_{N'}^t f \right\|_{L^2}^2 .
\end{aligned}$$

Then, using Lemma 3.3, we find that

$$\begin{aligned}
\left\| \Pi_N^u \mathcal{K}^n \sum_{\substack{N' \geq 0, t \in \{u,s\} \\ (N,u) \not\leftrightarrow (N',t)}} \Pi_{N'}^t f \right\|_{L^2} &\leq \sum_{\substack{N' \geq 0, t \in \{u,s\} \\ (N,u) \not\leftrightarrow (N',t)}} \left\| \Pi_N^u \mathcal{K}^n \Pi_{N'}^t \right\|_{L^2 \to L^2} \left\| \Pi_{N'}^t f \right\|_{L^2} \\
&\leq C_{n,L} 2^{LN} \sum_{\substack{N' \geq 0, t \in \{u,s\} \\ (N,u) \not\leftrightarrow (N',t)}} 4^{LN'} \left\| \Pi_{N'}^t f \right\|_{L^2} \\
&\leq C_{n,L} 2^{LN} \left(\sum_{\substack{N' \geq 0, t \in \{u,s\} \\ (N,u) \not\leftrightarrow (N',t)}} 4^{LN'} \right)^{\frac{1}{2}} \|f\|_{H^L} \\
&\leq \widetilde{C}_{n,L} 2^{LN} \|f\|_{H^L} .
\end{aligned}$$

Summing over $N \geq 0$, we find that

$$\begin{aligned}
\sum_{N\geq 0} 4^{\alpha N} \left\|\Pi_N^u \mathcal{K}^n f\right\|_{L^2}^2 &\leq 2 \sum_{N\geq 0} \|\mathcal{K}\|_{L^2\to L^2}^{2n} 4^{\alpha N} \sum_{N'\geq N+n\log_2\rho-\log_2 C_0-2} \left\|\Pi_{N'}^u f\right\|_{L^2}^2 \\
&\qquad + 2\sum_{N\geq 0} 4^{\alpha N} 4^{LN} \|f\|_{H^L}^2 \\
&\leq 2\|\mathcal{K}\|^{2n} \sum_{N'\geq 0} \left(\sum_{N\leq N'-n\log_2\rho+\log_2 C_0+2} 4^{\alpha N} \right) \left\|\Pi_{N'}^u f\right\|_{L^2}^2 \\
&\qquad + C_{n,L} \|f\|_{H^L}^2 \\
&\leq \frac{2C_0^{2\alpha} 64^\alpha}{4^\alpha - 1} \left(\|\mathcal{K}\|_{L^2\to L^2} \rho^{-\alpha}\right)^{2n} \sum_{N'\geq 0} 4^{\alpha N'} \left\|\Pi_{N'}^u f\right\|^2 + C_{n,L}\|f\|_{H^L}^2 \\
&\leq \frac{2C_0^{2\alpha} 64^\alpha}{4^\alpha - 1} \left(\|\mathcal{K}\|_{L^2\to L^2} \rho^{-\alpha}\right)^{2n} \|f\|_\alpha + C_{n,L}\|f\|_{H^L}^2 .
\end{aligned} \tag{20}$$

Here, we assumed (without loss of generality) that $L < -|\alpha|$.

Now, for $N \geq 0$, we find using Lemma 3.3 as above, that

$$\begin{aligned}
&\left\|\Pi_N^s \mathcal{K}^n f\right\|_{L^2}^2 \\
&\quad \leq 2\|\mathcal{K}\|_{L^2\to L^2}^{2n} \left(\sum_{0\leq N'\leq N-n\log_2\rho+\log_2 C_0+2} \left\|\Pi_{N'}^s f\right\|_{L^2}^2 \right. \\
&\qquad\qquad \left. + \sum_{\substack{N'\geq 0 \\ N'\geq n\log_2\rho \text{ or } N\geq n\log_2\rho}} \left\|\Pi_{N'}^u f\right\|_{L^2}^2 \right) \\
&\qquad\qquad + 4^{LN}\|f\|_{H^L}^2
\end{aligned}$$

Summing over N, we find that

$$\begin{aligned}
&\sum_{N\geq 0} 4^{-\alpha N} \left\|\Pi_N^s \mathcal{K}^n f\right\|_{L^2}^2 \\
&\quad \leq 2\|\mathcal{K}\|_{L^2\to L^2}^{2n} \sum_{N'\geq 0} \left(\sum_{N\geq N'+n\log_2\rho-\log_2 C_0-2} 4^{-\alpha N} \right) \left\|\Pi_{N'}^s f\right\|_{L^2}^2 \\
&\qquad + 2\|\mathcal{K}\|_{L^2\to L^2}^{2n} \sum_{N'\geq 0} \left(\sum_{N\geq n\log_2\rho} 4^{-\alpha N} \right) \left\|\Pi_{N'}^u f\right\|_{L^2}^2 \\
&\qquad + 2\|\mathcal{K}\|_{L^2\to L^2}^{2n} \sum_{N'\geq n\log_2\rho} \left(\sum_{N\geq 0} 4^{-\alpha N} \right) \left\|\Pi_{N'}^u f\right\|_{L^2}^2 \\
&\qquad + C_{n,L}\|f\|_{H^L}^2 \\
&\quad \leq \frac{2C_0^{2\alpha} 64^\alpha}{1-4^{-\alpha}} \left(\|\mathcal{K}\|_{L^2\to L^2} \rho^{-\alpha}\right)^{2n} \sum_{N'\geq 0} 4^{-\alpha N'} \left\|\Pi_{N'}^s f\right\|_{L^2}^2 \\
&\qquad + \frac{2}{1-4^{-\alpha}} \left(\|\mathcal{K}\|_{L^2\to L^2} \rho^{-\alpha}\right)^{2n} \sum_{N'\geq 0} \left\|\Pi_{N'}^u f\right\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{1-4^{-\alpha}}\left(\|\mathcal{K}\|_{L^2\to L^2}\,\rho^{-\alpha}\right)^{2n}\sum_{N'\geq 0}4^{\alpha N'}\,\|\Pi^u_{N'}f\|^2_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + C_{n,L}\,\|f\|^2_{H^L}\\
\leq\ &\frac{2}{1-4^{-\alpha}}\left(C_0^{2\alpha}64^{\alpha}+2\right)\left(\|\mathcal{K}\|_{L^2\to L^2}\,\rho^{-\alpha}\right)^{2n}\|f\|^2_{\mathcal{H}^{\alpha}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + C_{n,L}\,\|f\|^2_{H^L}\,. \quad (21)
\end{aligned}
$$

Now, take $\alpha_0 > 0$ large enough so that $\tau := \|\mathcal{K}\|^{\frac{1}{\alpha_0}}_{L^2\to L^2}\,\rho^{-1} < 1$, and sum (20) and (21) to get the result for $\alpha \geq \alpha_0$ (the estimate with $n = 1$ implies that $\mathcal{K}$ is bounded on $\mathcal{H}_\alpha$). □

Remark 3.4. Let $g \in C^\infty(\mathbb{T}^2)$. Notice that with the same proof as Lemma 3.2, we find that the operator $g\mathcal{K}$ also satisfies Lemma 3.2 (with the same value of ϵ). In particular, $g\mathcal{K}$ is bounded on $\mathcal{H}_\alpha$.

3.4 Quasi-compactness

Notice that taking $L < -\alpha$ in Lemma 3.2, the injection of $\mathcal{H}_\alpha$ in H^L is compact (because the injection of $H^{-|\alpha|}$ in H^L is). Then, Hennion's theorem [Hen93] based on Nussbaum's formula for essential spectral radius [Nus70] implies with Lemma 3.2 that:

Proposition 3.5. *Under the assumption of Lemma 3.2, the intersection of the spectrum of $\mathcal{K}$ acting on $\mathcal{H}_\alpha$ and $\{z \in \mathbb{C} : |z| > \tau^\alpha\}$ is made of isolated eigenvalues of finite multiplicity.*

More precisely, Hennion proves that if $\eta > \tau^\alpha$ then there is a decomposition of $\mathcal{H}_\alpha$ as

$$\mathcal{H}_\alpha = E \oplus F, \tag{22}$$

where E and F are invariant by $\mathcal{K}$, the space E is finite dimensional, F is closed and the operator induced by $\mathcal{K}$ on F has spectral radius strictly less than η. Let then A and B denotes the operators induced by $\mathcal{K}$ respectively on E and F. Let P_E and P_F denote the projectors respectively on E and F according to (22). If $g \in C^\infty(\mathbb{T}^2)$, let $P_E^* g$ denote the linear form on E obtained by restricting $f \mapsto \int_{\mathbb{T}^2} fg\mathrm{d}x$.

With these notation, if $f, g \in C^\infty(\mathbb{T}^2)$, we have for $n \geq 0$

$$\int_{\mathbb{T}^2} f\circ F^n g\mathrm{d}x = \int_{\mathbb{T}^2}\mathcal{K}^n(f)g\mathrm{d}x = P_E^* g(A^n P_E f) + \int_{\mathbb{T}^2} B^n fg\mathrm{d}x.$$

Since the spectral radius of B is strictly less than η, we find that

$$\int_{\mathbb{T}^2} B^n fg\mathrm{d}x \underset{n\to+\infty}{=} \mathcal{O}(\eta^n).$$

The asymptotics of the sequence $(P_E^* g(A^n P_E f))_{n\geq 0}$ is obtained by writing the Jordan decomposition of the finite dimensional linear operator A. Theorem 1 follows (since α can be taken arbitrarily large, η may be chosen arbitrarily small), except maybe the fact that the $a_{\lambda,k}(f,g)$'s are non-trivial, which is explained below. Notice also that Remark 1.6 implies that the resonances in Theorem 1 do not depend on the particular construction of the space $\mathcal{H}_\alpha$.

The decomposition (22) also implies that the resolvent $(z-\mathcal{K})^{-1}$, which is defined for $|z| \gg 1$ by

$$(z-\mathcal{K})^{-1} = \sum_{n \geq 0} z^{-(n+1)} \mathcal{K}^n,$$

has a meromorphic continuation to $\{z \in \mathbb{C} : |z| > \eta\}$ as an operator from $\mathcal{H}_\alpha$ to itself (and thus from $C^\infty(\mathbb{T}^2)$ to $\mathcal{D}'(\mathbb{T}^2)$). Indeed, $(z-A)^{-1}$ is meromorphic on $\mathbb{C}$, as given for instance by Cramer's formula, and $(z-B)^{-1}$ is holomorphic on $\{z \in \mathbb{C} : |z| > \eta\}$ since the spectral radius of B is less than η. Thus, we have

$$(z-\mathcal{K})^{-1} = (z-A)^{-1} P_E + (z-B)^{-1} P_F.$$

Once again, since η may be chosen arbitrarily small by taking α large, we find that $(z-\mathcal{K})^{-1}$ has a meromorphic extension to $\mathbb{C} \setminus \{0\}$ as an operator from $C^\infty(\mathbb{T}^2)$ to $\mathcal{D}'(\mathbb{T}^2)$. Moreover, notice that at any pole of $(z-\mathcal{K})^{-1}$ the coefficients of negative indexes in the Laurent expansion have finite rank.

Let us recall that if λ is an eigenvalue of $\mathcal{K}$ of modulus larger than η, then one may define a spectral projector on the eigenspace E_λ for $\mathcal{K}$ associated to λ in the following way [Kat95, §III.6.4-5]

$$P_\lambda = \frac{1}{2i\pi} \int_{\partial \mathbb{D}(\lambda, \epsilon)} (z-\mathcal{K})^{-1} \mathrm{d}z,$$

where $\epsilon > 0$ is small enough so that the closed disc of center λ and radius ϵ intersects the spectrum of $\mathcal{K}$ on $\mathcal{H}_\alpha$ only at λ. Then, this projector commutes with $\mathcal{K}$ and, with P_E defines as above, we have

$$P_E = \sum_{\substack{\lambda \in \sigma(\mathcal{K}_{|\mathcal{H}_\alpha}) \\ |\lambda| \geq \eta}} P_\lambda.$$

Moreover, the Laurent expansion of $(z-\mathcal{K})^{-1}$ is given near an isolated eigenvalue λ with $|\lambda| \geq \eta$ by [Kat95, §III.6.4-5]

$$(z-\mathcal{K})^{-1} = J(z) + \sum_{k=0}^{M(\lambda)} (-1)^k \frac{(\lambda-\mathcal{K})^k P_\lambda}{(z-\lambda)^{k+1}}$$

where $J(z)$ is holomorphic near λ and $M(\lambda)$ is the maximal size of a Jordan block of $\mathcal{K}$ at λ.

Notice that the meromorphic extension of the function $\Psi_{f,g}(z)$ from Remark 1.6 is given in term of the resolvent $(z-\mathcal{K})^{-1}$ by

$$\Psi_{f,g}(z) = \int_{\mathbb{T}^2} ((z-\mathcal{K})^{-1} f) g \mathrm{d}x.$$

Thus, for z near a resonance λ, we find that

$$\Psi_{f,g}(z) = \int_{\mathbb{T}^2} (J(z) f) g \mathrm{d}x + \sum_{k=0}^{M(\lambda)} \frac{1}{(z-\lambda)^{k+1}} \int_{\mathbb{T}^2} ((\lambda-\mathcal{K})^k P_\lambda f) g \mathrm{d}x.$$

Now, if $\ell \in \{0, \dots, M(\lambda)\}$, since $C^\infty(M)$ is dense in $\mathcal{H}_\alpha$ and P_λ has finite rank, we can find $f \in C^\infty(M)$ such that $(\lambda-\mathcal{K})^\ell P_\lambda f \neq 0$ and $(\lambda-\mathcal{K})^k P_\lambda f = 0$ for

$k = \ell+1, \dots, M(\lambda)$. Then, taking g such that $\int_{\mathbb{T}^2} (\lambda - \mathcal{K})^\ell P_\lambda f g \mathrm{d}x \neq 0$ and comparing with Remark 1.6, we find that the resonances of modulus larger than η in Theorem 1 are exactly the eigenvalues of modulus larger than η of $\mathcal{K}$ on $\mathcal{H}_\alpha$ (*a priori* there could have been less resonances than eigenvalues) and that $N(\lambda) = M(\lambda)$ for $\lambda \in \mathrm{Res}$. We also find that the $a_{\lambda,k}$'s are non-trivial.

3.5 Proof of quasi-compactness

In this section, we give a proof of Proposition 3.5 that relies on Fredholm analytic theory [DZ19, Theorem C.8] rather than on Hennion's argument [Nus70, Hen93].

Proof of Proposition 3.5. Let $z \in \mathbb{C}$ be such that $|z| > \tau^\alpha$. We are going to prove that $z - \mathcal{K}$ defines a semi-Fredholm operator on $\mathcal{H}_\alpha$. Pick some integer $m \geq 0$, and notice that for $f \in \mathcal{H}_\alpha$ we have

$$z^{-1} \left(\sum_{k=0}^{m-1} z^{-k} \mathcal{K}^k \right) (z - \mathcal{K}) f = f - z^{-m} \mathcal{K}^m f.$$

Thus

$$f = z^{-m}\mathcal{K}^m f + z^{-1} \left(\sum_{k=0}^{m-1} z^{-k} \mathcal{K}^k \right) (z - \mathcal{K}) f,$$

and it follows from Lemma 3.2 that (we take $L \ll -\alpha$)

$$\|f\|_\alpha \leq C|z|^{-m}\tau^{\alpha m} + C_{m,z} \|f\|_{H^L} + C_{m,z} \|(z-\mathcal{K})f\|_{\mathcal{H}_\alpha}.$$

Taking m large enough, we get $C|z|^{-m}\tau^{\alpha m} < 1$, which gives

$$\|f\|_\alpha \leq \frac{C_{m,z}}{1 - C|z|^{-m}\tau^{\alpha m}} \|f\|_{H^L} + \frac{C_{m,z}}{1 - C|z|^{-m}\tau^{\alpha m}} \|(z-\mathcal{K})f\|_{\mathcal{H}_\alpha}.$$

That is, for some new constant C that depends on z,

$$\|f\|_\alpha \leq C \|(z-\mathcal{K})f\|_\alpha + C \|f\|_{H^L} \text{ for every } f \in \mathcal{H}_\alpha. \tag{23}$$

Let us prove that $\ker(z - \mathcal{K})$ is finite dimensional. Let $(f_n)_{n\geq 0}$ be a bounded sequence in $\ker(z-\mathcal{K})$. Since the injection of $\mathcal{H}_\alpha$ in H^L is compact (because $L < -\alpha$), we can extract a subsequence $(f_{n_j})_{j\geq 0}$ that converges in H^L. It follows from (23) that $(f_{n_j})_{j \geq 0}$ is Cauchy in $\mathcal{H}_\alpha$ and thus converges to an element of $\ker(z-\mathcal{K})$ in $\mathcal{H}_\alpha$. Hence, bounded subsets of $\ker(z-\mathcal{K})$ are compact, and it follows from Riesz theorem that $\ker(z-\mathcal{K})$ is finite dimensional.

Let us prove now that the image of $z - \mathcal{K}$ is closed. Let $(g_n)_{n \geq 0}$ be a sequence of elements of the image of $z - \mathcal{K}$ that converges to some g in $\mathcal{H}_\alpha$. For each $n \geq 0$, choose $f_n \in \mathcal{H}_\alpha$ such that $g_n = (z-\mathcal{K})f_n$ and then $h_n \in \ker(z-\mathcal{K})$ such that $\|f_n - h_n\|_\alpha \leq (1 + 2^{-n}) d(f_n, \ker(z-\mathcal{K}))$.

If $(f_n - h_n)_{n\geq 0}$ has a bounded subsequence (in $\mathcal{H}_\alpha$), then one can extract a subsequence $(f_{n_j} - h_{n_j})_{j \geq 0}$ that converges in H^L. Since $(z-\mathcal{K})(f_{n_j} - h_{n_j}) = g_{n_j}$ converges in $\mathcal{H}_\alpha$, it follows from (23) that $(f_{n_j} - h_{n_j})_{j\geq 0}$ is Cauchy, and thus converges, in $\mathcal{H}_\alpha$. If f is the limit, we have $(z - K)f = g$ and thus g belongs to the image of $z - \mathcal{K}$.

If $(f_n - h_n)_{n\geq 0}$ has no bounded subsequence, then $\|f_n - h_n\|_\alpha \underset{n\to+\infty}{\to} \infty$. For n large enough, define $x_n = (f_n - h_n)/\|f_n - h_n\|_\alpha$. Then $(x_n)_{n\geq 0}$ is bounded in $\mathcal{H}_\alpha$. Extract a subsequence $(x_{n_j})_{j\geq 0}$ that converges in H^L. Notice that

$$(z-\mathcal{K})x_n = \frac{g_n}{\|f_n - h_n\|_\alpha} \underset{n\to+\infty}{\to} 0.$$

Thus, it follows from (23) that $(x_{n_j})_{j\geq 0}$ is Cauchy, and thus converges, in $\mathcal{H}_\alpha$. Let x be the limit, and notice that $(z-\mathcal{K})x = 0$. Thus, for $j \geq 0$, we have

$$\begin{aligned} d(f_{n_j}, \ker(z-\mathcal{K})) &\leq \left\| f_{n_j} - h_{n_j} - \left\|f_{n_j} - h_{n_j}\right\|_\alpha x\right\|_\alpha \\ &\leq \left\|f_{n_j} - h_{n_j}\right\|_\alpha \left\|x_{n_j} - x\right\|_\alpha \\ &\leq (1+2^{-n_j}) \left\|x_{n_j} - x\right\|_\alpha d(f_{n_j}, \ker(z-\mathcal{K})), \end{aligned}$$

which gives a contradiction for j large enough (recall that $\|f_n - h_n\|_\alpha \underset{n\to+\infty}{\to} \infty$).

Thus for $|z| > \tau^\alpha$, the operator $z - \mathcal{K}$ is semi-Fredholm on $\mathcal{H}_\alpha$. It follows from [Kat95, Theorem IV.5.17] that the index of $z-\mathcal{K}$ does not depend on z. Since $z - \mathcal{K}$ is invertible when $|z| \gg 1$, we find that the index of $z-\mathcal{K}$ is 0 when $|z| > \tau^\alpha$, and thus $z-\mathcal{K}$ is Fredholm. Hence, Proposition 3.5 follows from the analytic fredholm theory [DZ19, Theorem C.8]. $\square$

4 SRB measure

4.1 Regularity properties of anisotropic distributions

Introduce the cone

$$\mathcal{C}^u = \left\{ v \in \mathbb{R}^2 : \langle w, v\rangle \neq 0 \text{ for every } w \in \overline{\mathbb{R}^2 \setminus \mathcal{C}_*^u} \setminus \{0\} \right\}.$$

Proposition 4.1. *Let W be a C^∞ curve in $\mathbb{T}^2$. Assume that for every $x \in W$, the tangent space T_xW to W at x is contained in the cone $\mathcal{C}^u$. Then for $\alpha > 2$ the restriction operator $R_W : C^\infty(\mathbb{T}^2) \to C^\infty(W)$ extends to a continuous operator $R_W : \mathcal{H}_\alpha \to \mathcal{D}'(W)$.*

Remark 4.2. Let e_u and e_s be eigenvectors for A associated to the eigenvalues λ and $\pm\lambda^{-1}$. Notice that $\langle e_u, e_u^*\rangle = 0$ and thus $\langle e_u, e_s^*\rangle \neq 0$. Consequently, e_u is in the interior of $\mathcal{C}^u$. Hence any curve which is C^1 close to a piece of unstable manifold of F_0 satisfies the assumptions of Proposition 4.1. One can also check that if $\|\varphi\|_{C^1}$ is small enough then any piece of unstable manifold for F satisfies the assumptions from Proposition 4.1.

Remark 4.3. On $\mathbb{T}^2$, the presence of a reference measure allows us to identify densities and smooth functions, and thus the space $\mathcal{D}'(\mathbb{T}^2)$ of distributions on $\mathbb{T}^2$ with the dual of $C^\infty(\mathbb{T}^2)$. However, if W is a curve in $\mathbb{T}^2$ (as in Proposition 4.1 for instance), we will actually work with $\mathcal{D}'(W)$ as a space of functionals on the space of densities on W.

Proof of Proposition 4.1. Let $f \in \mathcal{H}_\alpha$. Notice that there is a constant $C > 0$ (that does not depend on f) such that, for every $k \in \mathbb{Z}^2$, we have

$$|c_k(f)| \leq C(1+|k|)^{-\alpha} \|f\|_\alpha \text{ if } k \in \mathcal{C}_*^u$$

and

$$|c_k(f)| \leq C(1+|k|)^{\alpha} \|f\|_{\alpha} \text{ if } k \notin \mathcal{C}^u_*.$$

Since $\alpha > 2$, we see that

$$\sum_{\substack{k \in \mathbb{Z}^2 \\ k \in \mathcal{C}^u_*}} c_k(f) e^{2i\pi k \cdot x} \tag{24}$$

converges to a bounded continuous function (on $\mathbb{T}^2$ and thus on W) whose supremum norm is controlled by $\|f\|_{\alpha}$.

Let K be a compact subset of W and φ a smooth 1-form on W supported in K. Let $\gamma : I \to \mathbb{T}^2$ be a parametrization of W by arclength (with I an interval of $\mathbb{R}$). Let $J = \gamma^{-1}(K)$, and notice that J is compact. Write $\gamma^* \varphi = g \mathrm{d}t$ with g a smooth function supported in J. From our assumption on W, we see that there is a constant $C > 0$ such that

$$|\langle w, \gamma'(t) \rangle| \geq C^{-1} |w| \text{ for every } w \in \mathcal{C}^u_* \text{ and } t \in J. \tag{25}$$

For $k \in \mathbb{Z}^2 \setminus \mathbb{C}^u_*$, we have

$$\int_W e^{2i\pi k \cdot x} \varphi = \int_I e^{2i\pi k \cdot \gamma(t)} g(t) \mathrm{d}t.$$

Letting L be the differential operator on I defined by

$$Lu = -\left(\frac{u}{2i\pi k \cdot \gamma'} \right)',$$

we find, integrating by part, that (for any $m \geq 0$)

$$\int_W e^{2i\pi k \cdot x} \varphi = \int_I e^{2i\pi k \cdot \gamma(t)} L^m g(t) \mathrm{d}t.$$

By induction, we find that $L^m g$ is a linear combination of terms of the form

$$\frac{(k \cdot \gamma^{(\ell_1)}(t)) \dots (k \cdot \gamma^{(\ell_p)}(t)) g^{(q)}}{(\gamma'(t))^{m+p}},$$

where q is less than m. Thus, from (25), there is a constant $C > 0$ such that

$$\left| \int_W e^{2i\pi k \cdot x} \varphi \right| \leq C(1+|k|)^{-m} \|g\|_{C^m}$$

for every $k \in \mathbb{Z}^2 \setminus \mathcal{C}^u_*$. Taking $|k| \geq \alpha + 2$, we find that the sum

$$\sum_{\substack{k \in \mathbb{Z}^2 \\ k \in \mathcal{C}^u_*}} c_k(f) e^{2i\pi k \cdot x} \tag{26}$$

converges to a distribution on W of order $\alpha + 2$.

We can finally define $R_W f$ as the sum of the distributions (24) and (26). It follows from the estimate above that R_W is bounded from $\mathcal{H}_{\alpha}$ to $\mathcal{D}'(W)$. If f is a C^{∞} function then $R_W f$ is the restriction of f to W since f is the sum of its Fourier series. □

Remark 4.4. One can check that if $f \in \mathcal{H}_\alpha$ then $R_W f$ depends continuously on W in the following sense. Let I be an interval of $\mathbb{R}$ and $(\gamma_n)_{n \geq 0}, \gamma$ be C^∞ embeddings from I to $\mathbb{T}^2$. Assume that $W = \gamma(I)$ and $W_n = \gamma_n(I), n \geq 0$ satisfy the assumptions from Proposition 4.1, and that for each $k \geq 0$ the function $\gamma_n^{(k)}$ converges to $\gamma^{(k)}$ when n goes to $+\infty$ uniformly on all compact subsets of I. Let φ be a smooth 1-form on $\mathbb{T}^2$ such that there is a compact subset K of I such that $\gamma_n^* \varphi$ is supported in K for every $n \geq 0$. Then

$$\int_{W_n} R_{W_n} f \varphi \underset{n \to +\infty}{\to} \int_W R_W \varphi.$$

Indeed, this convergence holds when $f \in C^\infty(\mathbb{T}^2)$. Moreover, it follows from the proof of Proposition 4.1 that the linear forms

$$f \mapsto \int_W R_W f \varphi - \int_{W_n} R_{W_n} f \varphi,$$

for $n \geq 0$, are uniformly bounded on $\mathcal{H}_\alpha$. The result then follows by an approximation argument.

4.2 Understanding the largest eigenvalue

We are now going to study the largest eigenvalues of the operator $\mathcal{K}$ acting on the space $\mathcal{H}^\alpha$, and prove Theorem 2 in the particular case of the Anosov diffeomorphisms from §3.1.

In all this section, we assume that the assumptions from Proposition 3.5 are satisfied and choose α large enough so that Proposition 4.1 holds. We split the proof in several parts of different lengths.

First, notice that 1 is a resonance for $\mathcal{K}$. Indeed, $\mathcal{K}1 = 1$ and the function constant equal to 1 belongs to $\mathcal{H}_\alpha$ since it is C^∞. Recall that we proved in §3.4 that the eigenvalues of modulus strictly larger than τ^α of $\mathcal{K}$ on $\mathcal{H}_\alpha$ are resonances.

Lemma 4.5. *Under the assumption of Proposition 3.5, there is no resonance of modulus larger than* 1*, and the eigenvalues of* $\mathcal{K}$ *on* $\mathcal{H}_\alpha$ *of modulus* 1 *have no Jordan block.*

Proof. The idea of the proof is relatively simple: a resonance of modulus strictly larger than 1 or a resonance of modulus 1 with a Jordan block would produce an unbounded term in the right hand side of (1), while the left hand side is clearly bounded. The actual proof is slightly more complicated since we need to explain why there cannot be cancellation between the different terms in the right hand side of (1) that would make the sum bounded despite the presence of unbounded terms.

Let ρ be the maximal modulus of a resonance (notice that $\rho \geq 1$). Let $\lambda_1, \dots, \lambda_d$ be the resonances of modulus ρ. Remember that $\tau^\alpha < 1 \leq \rho$. Recall $P_{\lambda_1}, \dots, P_{\lambda_d}$, the spectral projectors associated to $\lambda_1, \dots, \lambda_d$. For $j = 1, \dots, d$, since λ_j is the only eigenvalue of the operator induced by $\mathcal{K}$ on the image of P_{λ_j}, there is a nilpotent operator N_j on the image of P_{λ_j} such that $\mathcal{K} P_{\lambda_j} = (\lambda_j + N_j) P_{\lambda_j}$. Thus, we can write

$$\mathcal{K} = \sum_{j=1}^{d} (\lambda_j + N_j) P_{\lambda_j} + Q,$$

where Q is a bounded operator on $\mathcal{H}_\alpha$ with spectral radius strictly less than ρ such that $QP_{\lambda_1} = \dots = QP_{\lambda_d} = P_{\lambda_1}Q = \dots = P_{\lambda_d}Q = 0$. Letting k_j be the smallest integer such that $N_j^{k_j} = 0$, we find that for $n \geq 0$, we have

$$\mathcal{K}^n = \sum_{j=1}^{d} \sum_{\ell=0}^{k_j-1} \begin{pmatrix} n \\ \ell \end{pmatrix} \lambda_j^{n-\ell} N_j^\ell P_{\lambda_j} + Q^n.$$

Without loss of generality, we may assume that k_1 is larger than or equal to $k_2, \dots, k_d$. Then, we find that

$$\lim_{n \to +\infty} \frac{1}{n^{k_1}} \sum_{m=0}^{n-1} \lambda_1^{-m} \mathcal{K}^m = \frac{\lambda_1^{1-k_1}}{k_1!} N_1^{k_1-1} P_{\lambda_1} \tag{27}$$

as bounded operator on $\mathcal{H}_\alpha$. Since $N_1^{k_1-1} \neq 0$ by definition of k_1, there is $h \in \mathcal{H}_\alpha$ in the image of P_{λ_1} such that $N_1^{k_1-1}h \neq 0$. Since $C^\infty(M)$ is dense in $\mathcal{H}_\alpha$ and P_{λ_1} has finite rank, there is $f \in C^\infty(M)$ such that $P_{\lambda_1} f = h$. Let now $g \in C^\infty(M)$ be such that $\int_{\mathbb{T}^2} N_1^{k_1-1} h g \mathrm{d}x \neq 0$.

It follows then from (27) that

$$\frac{\lambda_1^{1-k_1}}{k_1!} \int_{\mathbb{T}^2} N_1^{k_1-1} h g \mathrm{d}x = \lim_{n \to +\infty} \frac{1}{n^{k_1}} \sum_{m=0}^{n-1} \lambda_1^{-m} \int_{\mathbb{T}^2} f \circ T^m g \mathrm{d}x.$$

Notice then that for $n \geq 0$, we have

$$\left| \frac{1}{n^{k_1}} \sum_{m=0}^{n-1} \lambda_1^{-m} \int_{\mathbb{T}^2} f \circ T^m g \mathrm{d}x \right| \leq \frac{1}{n^{k_1}} |f|_\infty \int_{\mathbb{T}^2} |g| \mathrm{d}x \sum_{m=0}^{n-1} \rho^{-m}. \tag{28}$$

If $\rho > 1$, we see that this quantity goes to 0 as n goes to ∞, which contradicts $\int_{\mathbb{T}^2} N_1^{k_1-1} h g \mathrm{d}x \neq 0$.

Thus $\rho = 1$. Similarly, if $k_1 \geq 2$, then (28) implies that $\int_{\mathbb{T}^2} N_1^{k_1-1} h g \mathrm{d}x = 0$, which is a contradiction. Thus $k_1 = k_2 = \dots = k_d = 1$, that is there is no Jordan block for resonances of modulus 1. □

Lemma 4.6. *Under the assumption of Proposition 3.5, if F is transitive, then 1 is a simple eigenvalue of $\mathcal{K}$ on $\mathcal{H}_\alpha$ and there is no other eigenvalue of modulus 1.*

Remark 4.7. Notice that the transitivity assumption in Lemma 4.6 (and in Lemma 4.8 below) is actually not needed. Indeed, it follows from the structural stability of hyperbolic CAT maps [KH95, Theorem 2.6.3] that the Anosov diffeomorphisms from §3.1 are transitive. We will also use in the proof of Lemma 4.6 the fact that $\mathbb{T}^2$ is connected.

Proof of Lemma 4.6. The proof of this lemma is adapted from the proof of [GL08, Theorem 5.1].

Let λ be a resonance of $\mathcal{K}$ with modulus 1 and let $h \in \mathcal{H}_\alpha$ be an eigenvector associated to λ. Let P_λ be the spectral projector associated to λ. Since $C^\infty(M)$ is dense in $\mathcal{H}_\alpha$ and P_λ has finite rank, there is $f \in C^\infty(M)$ such that $P_\lambda f = h$.

Working as in the proof of Lemma 4.6, we find that

$$P_\lambda = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} \mathcal{K}^k,$$

and it follows that for every $g \in C^\infty(\mathbb{T}^2)$ we have

$$\left| \int_{\mathbb{T}^2} hg \mathrm{d}x \right| \leq |f|_\infty \int_{\mathbb{T}^2} |g| \mathrm{d}x. \tag{29}$$

Thus, h is a L^∞ function.

Let now W be a piece of unstable manifold for F, and recall that W satisfies the assumptions from Proposition 4.1. Let $v \in \mathbb{R}^2$ be transverse to W at every point of W (such a v exists because W satisfies the assumption from Proposition 4.1). Let $\chi : \mathbb{R} \to [0, \infty)$ be a compactly supported function of integral 1 and let $\chi_\epsilon : x \mapsto \epsilon^{-1} \chi(\epsilon^{-1} x)$ for $\epsilon > 0$. Notice that if $\delta > 0$ is small enough then the map

$$\begin{array}{cccc} G & : & W \times (-\delta, \delta) & \to & \mathbb{T}^2 \\ & & (x,t) & \mapsto & x + tv \end{array}$$

is a diffeomorphism on its image (which is a neighbourhood of W in $\mathbb{T}^2$). Let now $\varphi \in \mathcal{D}(W)$. For $t \in]-\delta, \delta[$, let φ_t denote the pullback of φ by $x \mapsto x - tv$, this is a density on $W + tv$. It follows from Remark 4.4 that

$$\int_W R_W h \varphi = \lim_{\epsilon \to 0} \int_{-\delta}^{\delta} \chi_\epsilon(t) \left(\int_{W+tv} R_{W+tv} h \varphi_t \right) \mathrm{d}t.$$

Let m_W denotes the arclength measure on W and write $\varphi = \psi \mathrm{d}m_W$ where ψ is C^∞ and compactly supported. Then, for $\epsilon > 0$ small enough, we have

$$\int_{-\delta}^{\delta} \chi_\epsilon(t) \left(\int_{W+tv} R_{W+tv} h \varphi_t \right) \mathrm{d}t = \int_{\mathbb{T}^2} h (\chi_\epsilon \psi) \circ G^{-1} J_{G^{-1}} \mathrm{d}x, \tag{30}$$

where $J_{G^{-1}}$ denotes the Jacobian of G^{-1} (where $W \times (-\delta, \delta)$ is endowed with the product measure). Indeed, for a smooth function h the formula (30) is a consequence of the change of variable formula, and the general case follows by a density argument (since both sides are continuous linear forms on $\mathcal{H}_\alpha$). Using (29), the change of variable formula again and Fubini's theorem, we find that

$$\begin{aligned} \left| \int_{-\delta}^{\delta} \chi_\epsilon(t) \left(\int_{W+tv} R_{W+tv} h \varphi_t \right) \mathrm{d}t \right| &\leq |f|_\infty \int_{\mathbb{T}^2} \left| (\chi_\epsilon \psi) \circ G^{-1} \right| J_{G^{-1}} \mathrm{d}x \\ &\leq |f|_\infty \int_{-\delta}^{\delta} \chi_\epsilon(t) \mathrm{d}t \times \int_W |\varphi| = |f|_\infty \int_W |\varphi|. \end{aligned}$$

It follows that

$$\left| \int_W R_W h \varphi \right| \leq |f|_\infty \int_W |\varphi|,$$

and thus $R_W h$ is a L^∞ function, bounded by $|f|_\infty$.

Applying Lebesgue differentiation theorem to $R_W f$, we find that there is $x_0 \in W$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{W_\epsilon(x_0)} |R_W h(x_0) - R_W h(y)| \mathrm{d}m_W(y) = 0, \tag{31}$$

where $W_\epsilon(x_0)$ denotes the interval of arclength ϵ centered at x_0 in W.

Notice that for every $n \geq 0$, the curve $F^n W$ is a piece of unstable manifold, and thus satisfies the assumptions from Proposition 4.1. Moreove, we have

$$R_{F^n W} h \circ F^n = R_W h. \tag{32}$$

Indeed, this relation is immediate when h is smooth and both sides define continuous operators from $\mathcal{H}_\alpha$ to $\mathcal{D}'(W)$.

It follows from the unstable manifold theorem (see Proposition 1.4) that $F^n W$ becomes longer and longer when n goes to $+\infty$. Fix some small $\sigma > 0$, and choose a sequence $(\epsilon_n)_{n\geq 0}$ going to 0 such that for every $n \geq 0$ large enough the length of $F^n W_{\epsilon_n}(x_0)$ is σ. We can then find a strictly increasing sequence $(n_j)_{j\geq 0}$ of integers such that

- $\lambda^{n_j} \underset{j\to+\infty}{\to} 1$;
- $(F^{n_j}(x_0))_{j\geq 0}$ converges to a point $x \in \mathbb{T}^2$.

Then, it follows from Proposition 1.4 that $F^{n_j}(W_{\epsilon_{n_j}}(x_0))$ converges to a piece W_0 of the unstable manifold of x in the sense of Remark 4.4.

Let us prove that $R_{W_0} h$ is constant. Let $\varphi \in \mathcal{D}(W_0)$. Extend φ to a smooth 1-form on $\mathbb{T}^2$, and notice that

$$\begin{aligned}
&\left| \int_{W_0} R_{W_0} h\varphi - R_W h(x_0) \int_{W_0} \varphi \right| \\
&\qquad \leq \left| \int_{W_0} R_{W_0} h\varphi - \int_{F^{n_j} W_{\epsilon_{n_j}}(x_0)} R_{F^{n_j} W_{\epsilon_{n_j}}(x_0)} h\varphi \right| \\
&\qquad\quad + |\lambda^{n_j} - 1| \left| \int_{F^{n_j} W_{\epsilon_{n_j}}(x_0)} R_{W_{\epsilon_{n_j}}(x_0)} h \circ F^{n_j} \varphi \right| \\
&\qquad\quad + |R_W h(x_0)| \left| \int_{W_0} \varphi - \int_{F^{n_j} W_{\epsilon_{n_j}}(x_0)} \varphi \right| \\
&\qquad\quad + \int_{W_{\epsilon_{n_j}}(x_0)} |R_W h - R_W h(x_0)| |(F^{n_j})^* \varphi|.
\end{aligned} \tag{33}$$

Notice that we used (32) and the fact that $\mathcal{K}h = \lambda h$. Let us explain why all the terms in the right hand side of (33) go to 0 as j goes to ∞. The first one goes to 0 because of Remark 4.4. The seconde one because λ^{n_j} goes to 1 (the other factor is uniformly bounded since we know that $R_{W_{\epsilon_{n_j}}(x_0)} h \circ F^{n_j}$ is bounded by $|f|_\infty$). The third one goes to 0 because φ is continuous. The fourth term goes to 0 as can be seen from (31) and a bounded distortion argument.

Thus, we find $R_{W_0} h$ is constant on W_0. Since F is transitive, we can find a point with dense forward orbit close enough to x so that its stable manifold intersects W_0. Thus, there is a point $x' \in W_0$ whose forward image is dense. Then, working as above, we find that for every $y \in \mathbb{T}^2$, the restriction of h to a piece of unstable manifold of y is constant. Indeed, we can first take $(n_j)_{j\geq 0}$ such that $F^{n_j}(x') \underset{j\to+\infty}{\to} y$, and then, up to extraction assume that $(\lambda_{n_j})_{j\geq 0}$ converges (not necessarily to 1). We can then redo the argument above starting at x' instead of x_0 (the estimate (31) holds now because $R_{W_0} h$) is constant.

Let us now defined a function $\tilde{h}$ on $\mathbb{T}^2$ in the following way: for every $y \in \mathbb{T}^2$, we let $\tilde{h}(y)$ be the constant value of the restriction of h to a piece of the unstable manifold of y. It follows from Remark 4.4 that $\tilde{h}$ is a continuous function. Moreover, one can check that $\tilde{h}$ is a representative of h using Fubini's theorem and a density argument.

Let us prove that $\tilde{h}$ is constant. It follows from the definition of $\tilde{h}$ that it is locally constant (and thus constant) on each unstable manifold. Moreover, if y_0, y_1 belong to the same stable manifold then $d(F^n x, F^n y) \underset{n\to+\infty}{\to} 0$. Since for $n \geq 0$, we have

$$|\tilde{h}(y_0) - \tilde{h}(y_1)| = |\lambda^{-n}\tilde{h}(F^n y_0) - \lambda^{-n}\tilde{h}(F^n y_1)| = |\tilde{h}(F^n y_0) - \tilde{h}(F^n y_1)|.$$

It follows from the uniform continuity of $\tilde{h}$ that $\tilde{h}(y_0) = \tilde{h}(y_1)$. Now, if $x, y \in \mathbb{T}^2$ are close enough to each other, we find that $\tilde{h}(x) = \tilde{h}([x,y]) = \tilde{h}(y)$. Hence, $\tilde{h}$ is locally constant, and thus constant since $\mathbb{T}^2$ is connected.

Thus, we have $\lambda = 1$. Since there is no Jordan block and constant functions are the only eigenvectors, we find that 1 is a simple eigenvalue.

$\square$

Lemma 4.8. *Under the assumption of Proposition 3.5, if F is transitive then the left eigenvector of $\mathcal{K}$ associated to 1 is the SRB measure of F.*

Proof. Let l denote the left eigenvector of $\mathcal{K}$ associated to 1 such that $\ell(1) = 1$. Notice that there is an operator $Q : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ of spectral radius strictly less than 1 such that $\ell \circ Q = 0, Q(1) = 0$ and

$$\mathcal{K}f = l(f)1 + Qf \text{ for } f \in \mathcal{H}_\alpha. \tag{34}$$

In particular, for $n \geq 0$ and $f \in C^\infty(\mathbb{T}^2)$, we have

$$\mathcal{K}^n f = l(f)1 + Q^n f. \tag{35}$$

Thus, if $g \in \mathcal{C}^\infty(\mathbb{T}^2)$ and $\eta \in (0,1)$ is strictly larger than the spectral radius of Q, we have

$$\int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}x \underset{n\to+\infty}{=} l(f) \int_{\mathbb{T}^2} g \mathrm{d}x + \mathcal{O}(\eta^n).$$

Taking $g = 1$, since

$$\left| \int_{\mathbb{T}^2} f \circ F^n \mathrm{d}x \right| \leq |f|_\infty$$

for every $n \geq 0$, we find that $|l(f)| \leq |f|_\infty$. It implies that there is a complex Borel measure μ on $\mathbb{T}^2$ such that

$$l(f) = \int_{\mathbb{T}^2} f \mathrm{d}\mu \text{ for } f \in C^\infty(\mathbb{T}^2).$$

Moreover, we know that $\mu(\mathbb{T}^2) = l(1) = 1$ and that $|\mu|(\mathbb{T}^2) \leq 1$. Hence, μ is a probability measure. Since for $f \in C^\infty(M)$, we have

$$\int_{\mathbb{T}^2} f \mathrm{d}\mu = \lim_{n\to+\infty} \int_{\mathbb{T}^2} f \circ F^n \mathrm{d}x,$$

we find that μ is F-invariant. In order to prove that μ is the SRB measure, we will check that μ is physical, following an argument that can be found in [BG09, Appendix B].

Let $f \in C^\infty(\mathbb{T}^2)$. We want to prove that

$$B_f = \left\{ x \in \mathbb{T}^2 : \frac{1}{n} \sum_{k=0}^{n-1} f \circ F^k(x) \underset{n \to +\infty}{\to} \int_{\mathbb{T}^2} f \mathrm{d}\mu \right\}$$

has full Lebesgue measure. Without loss of generality, we may assume that $\int_{\mathbb{T}^2} f \mathrm{d}\mu = 0$ and that f is real-valued. Let us write $S_n f = \sum_{k=0}^{n-1} f \circ F^k$ for $n \geq 0$.

Using (34) we find that for $k, \ell \geq 0$ we have

$$f \circ F^k f \circ F^{k+\ell} = \mathcal{K}^k (f \mathcal{K}^\ell f) = \int_{\mathbb{T}^2} f \mathcal{K} Q^{\ell - 1} f \mathrm{d}\mu + Q^k (f \mathcal{K} Q^{\ell-1} f).$$

Since $f\mathcal{K}$ is bounded on $\mathcal{H}_\alpha$ (see Remark 3.4), we find that $f \circ F^k f \circ F^{k+\ell}$ is bounded in $\mathcal{H}_\alpha$ by some $C(\eta^k + \eta^{k+\ell})$ where $\eta \in (0,1)$ and C depends on f but not on k and ℓ. Thus

$$\begin{aligned} \left| \int_{\mathbb{T}^2} S_n f S_m f \mathrm{d}x \right| &\leq \sum_{k=0}^{n-1} \sum_{\ell=0}^{m-1} \left| \int_{\mathbb{T}^2} f \circ F^k f \circ F^\ell \mathrm{d}x \right| \leq C \sum_{k=0}^{n-1} \sum_{\ell=0}^{m-1} (\eta^k + \eta^\ell) \\ &\leq \widetilde{C}(m+n). \end{aligned}$$

Thus, expanding and using the estimate above to bound the rectangle terms, we find that, for some constant $C > 0$, we have

$$\int_{\mathbb{T}^2} \left| \frac{S_n f}{n} - \frac{S_m f}{m} \right|^2 \mathrm{d}x \leq C \left(\frac{1}{m} + \frac{1}{n} \right).$$

For $p \geq 1$, define then $g_p = S_{p^4} f / p^4$, and notice that $\|g_{p+1} - g_p\|_{L^2} \leq C p^{-2}$ for some constant $C > 0$ that does not depend on p. It follows from Borel–Cantelli Lemma that $(g_p)_{p \geq 0}$ converges Lebesgue almost everywhere to some function g. Using that f is bounded, we find that for $p \geq 1$ and $p^4 \leq n < (p+1)^4$ we have $|S_n f / n - g_p| \underset{p \to +\infty}{=} \mathcal{O}(p^{-1})$. It follows that $S_n f / n$ converges Lebesgue almost everywhere to g. Since the functions $(S_n f/n)_{n \geq 0}$ are uniformly bounded, we find that g is bounded and the dominated convergence theorem implies that $(S_n f/n)_{n \geq 0}$ converges to g in distribution. However, it follows from (35) that $(S_n f/n)_{n \geq 0}$ converges to $l(f) = \int_{\mathbb{T}^2} f \mathrm{d}\mu = 0$ in distribution and thus $g = 0$ Lebesgue almost everywhere. This proves that B_f has full Lebesgue measure.

Let now $(f_n)_{n \geq 0}$ be a sequence of elements of $C^\infty(\mathbb{T}^2)$ with dense image in $C^0(\mathbb{T}^2)$. Notice that $\bigcap_{n \geq 0} B_{f_n}$ has full Lebesgue measure. By an approximation argument we find that if $x \in \bigcap_{n \geq 0} B_{f_n}$ and $w : \mathbb{T}^2 \to \mathbb{C}$ is continuous then

$$\frac{1}{n} \sum_{k=0}^{n-1} w \circ F^k(x) \underset{n \to +\infty}{\to} \int_{\mathbb{T}^2} w \mathrm{d}\mu.$$

Hence μ is physical, and is consequently the SRB measure of F. $\square$

4.3 Exponential mixing of the SRB measure

Theorem 1 gives the asymptotics of correlations of F with respect to a smooth measure on M. However, one can use the same method to study the asymptotics of correlations with respect to the SRB measure μ.

Theorem 3. *Under the assumption of Theorem 1, for every $\lambda \in \text{Res}$ and $k \in 0, \ldots, N(\lambda)$, there is a continuous linear form $b_{\lambda,k} : C^\infty(M) \times C^\infty(M) \to \mathbb{C}$ that factorizes through a finite dimensional space, such that for every $\eta > 0$ and $f, g \in C^\infty(M)$ we have*

$$\int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}\mu \underset{n \to +\infty}{=} \sum_{\substack{\lambda \in \text{Res} \\ |\lambda| \geq \eta}} \sum_{k=0}^{N(\lambda)} b_{\lambda,k}(f,g) n^k \lambda^n + \mathcal{O}(\eta^n).$$

Moreover, if M is conenected,

$$b_{1,0} : (f,g) \mapsto \int_{\mathbb{T}^2} f \mathrm{d}\mu \int_{\mathbb{T}^2} g \mathrm{d}\mu.$$

Proof. Let us write the proof under the assumption of Proposition 3.5.

Let $f, g \in C^\infty(\mathbb{T}^2)$. Recall the left eigenvector l for $\mathcal{K}$ associated to the eigenvalue 1, given for $h \in C^\infty(\mathbb{T}^2)$ by

$$l(h) = \int_{\mathbb{T}^2} h \mathrm{d}\mu.$$

Notice that l is a bounded linear form on $\mathcal{H}_\alpha$, where α is large enough, so that Propositions 3.5 and 4.1 hold.

From Remark 3.4, we know that the operator $g\mathcal{K}$ is bounded on $\mathcal{H}_\alpha$. Thus for $n \geq 1$, we have

$$\int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}\mu l(g\mathcal{K}(\mathcal{K}^{n-1} f)),$$

and the result follows then from linear algebra using Proposition 3.5 as in §3.4. □

Corollary 4.9. *Under the assumption of Theorem 1, for every $f, g \in C^\infty(M)$, we have*

$$\int_{\mathbb{T}^2} f \circ F^n g \mathrm{d}\mu \underset{n \to +\infty}{\to} \int_{\mathbb{T}^2} f \mathrm{d}\mu \int_{\mathbb{T}^2} g \mathrm{d}\mu.$$

Remark 4.10. Notice that Corollary 4.9 implies that the SRB measure μ is mixing for F.

5 Going further

A natural question for the reader at the end of this minicourse is: how to generalize the proofs that were given in these notes to general smooth transitive Anosov diffeomorphisms? First, the generalization to the higher dimension is relatively simple: one can work with Fourier series on higher dimensional tori. Then, one need to get rid of the assumption that F is a small perturbation of a CAT map. To do so, one just needs to notice that, locally, any smooth map is a small perturbation of a linear map, and then use the space $\mathcal{H}_\alpha$ from these notes as a local model. To get a global space suited to any Anosov diffeomorphism just needs to glue together different copies of the local model, which can be done using a partition of unity. Finally, we totally ignored here the finitely differentiable case, in which the proof of a Lasota–Yorke inequality become more technical. The strategies that allow to deal with this difficulty with the kind of spaces that we used here are described for instance in [Bal18].

Let us give now a few references that are potential entry doors to the literature on hyperbolic dynamics using spaces of anisotropic distributions. There are plenty of possible constructions of spaces of anisotropic distributions, a panorama of these different methods is given in [Bal17]. We gave in this minicourse a pedestrian approach of the spaces constructed via microlocal analysis. A more elaborated exposition may be found in [FRS08, Bal18] in the case of Anosov diffeomorphisms or in [FS11, DZ16] for flows. For more geometric constructions, one can refer to [GL06, GL08, BL07, BL13]. More introductory expositions may be found in [Liv19, DKL21, Dem18]. Notice that the latter focuses on spaces that allow to study hyperbolic dynamical systems with some singularities (for more advanced references on this topic, see for instance [DL08, DZ14, DZ11, DZ13]).

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